PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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Theories of Lunar Libration [and Discussion]

Alan H. Cook and J. Kovalevsky

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Theories of lunar libration

By Alan H. Cook, F.R.S.
Cavendish Laboratory, Madingley Road, Cambridge

The measured distance between a point on the Moon and an observatory on the Earth varies with the librational motion of the Moon about her centre of mass. The motion is caused by the varying attraction of the Earth, Sun and planets upon the Moon and obeys highly nonlinear equations of motion. Because of the high precision of measurements with lunar laser ranging systems, the theory of the motion must be worked out in great detail and the absence of adequate developments limits the interpretation of lunar ranging observations. Numerical integration of the equations of motion is carried out at the Jet Propulsion Laboratory and Eckhardt has developed a semi-literal theory in which coefficients of periodic terms are calculated numerically. There is still need, however, for a literal theory. A brief account will be given of a new literal theory, the algebraic manipulations for which are being carried out by the Camal machine algebraic program developed in the Computer Laboratory at Cambridge. The third harmonic terms in the gravitational potential of the Moon are included and it is intended to include the effect of the Sun.

1. Introduction

The Moon, to a first approximation, moves round the Earth in a circular orbit that is coplanar with her equator and with the ecliptic, and she rotates about her principal axis of greatest inertia at the same angular velocity as that with which she describes her orbit, so presenting always the same face to the Earth. Were those conditions exactly satisfied, the distance of a point on the Moon from one on the Earth would change only with the axial rotation of the Earth; the conditions are not exactly satisfied and so distances between points on the Earth and on the Moon change with the position of the Moon in her orbit, both through the variation of the distance of the centres of masses because the Moon's orbit is elliptical, and through the rotation of the Moon about her axes of inertia relative to the direction of the Earth. Parts of the relative rotations come from the motion of the Moon about the Earth at an angular velocity that varies around the orbit on account of the eccentricity of the orbit and its inclination to the equator of the Moon; those are the *geometrical librations*. In addition, the Earth, as it oscillates about its mean position relative to axes fixed in the Moon, exerts torques on the Moon which produce rotations of the Moon about the mean positions of its principal axes; those are the *physical librations*. The calculation of the geometrical librations is a matter of straightforward geometry; this paper is concerned with the more involved matter of the physical librations.

The librations of the Moon have long been known from telescopic observations of the Moon, although the angular motion of a point on the surface of the Moon as seen from the Earth is only some 6″. Thus, observations of the librations were relatively crude until the advent of laser ranging to the Moon, but a motion of 6″ at the distance of the Moon is some 4 km, a large change compared with the resolution of a few centimetres expected of lunar laser ranging observations. While then a fairly crude theory suffices for interpretation of the telescopic observations, one including very many periodic terms calculated with high precision is needed to interpret laser ranging data. The theory is required for two purposes: to enable the librational terms to be removed from measured ranges so as to facilitate the determination of other types of term, and to enable estimates of the dynamical parameters of the Moon to be made. The moments of inertia of the Moon are of considerable importance to the physical study of the Moon because they set limits on possible distributions of mass with the Moon, and they are estimated by combining coefficients of second harmonic terms in the gravitational potential of the Moon, which are found from changes in the orbits of lunar satellites, with the relative changes of the moments of inertia that control the physical librations.

Whereas the largest term in the physical libration has an amplitude of a few hundredths of a radian, the resolution of ranging observations corresponds to rotations of the Moon of 10^{-8} rad. Thus, a precision of the order of 10^{-9} should be sought in any theory if it is to be adequate. So far as the first purpose of a theory goes – the prediction of librations so that they may be removed from observed ranges – numerical integration may be used, and the Jet Propulsion Laboratory provides numerical integrations for that purpose. Numerical integration by itself is well known not to be sufficient. In the first place, errors may accumulate over long times and should be checked against a formal theory. In the second place it is easier to see from a formal theory the types of periodic term that will occur, information that may be essential in the planning of programmes of observation. Lastly, the determination of the dynamical parameters of the Moon is greatly simplified if a formal theory is available. Eckhardt (1970) has produced a semi-literal theory which goes a long way to meeting those needs but the coefficients of periodic terms appear as numbers calculated from postulated values of the various dynamical parameters, and it remains desirable to have a formal theory with explicit polynomial expressions for the coefficients of periodic terms showing their dependence upon the dynamical parameters. Following some general remarks upon the nature of theories of libration I give a brief account of work in progress to develop an algebraic theory of the physical librations.

2. Forms of theories of the physical librations

An obvious way of calculating the physical librations is to use Euler's equations that equate the rates of change of angular momentum about the centre of mass of a body to the torques exerted upon it. Hayn (1902, 1923) reduces the Euler equations to the following form:

$$\dot{p}+\alpha qr = 3M\alpha yz/R^5,$$
$$\dot{q}-\beta pr = -3M\beta zx/R^5,$$
$$\dot{r}+\gamma pq = 3M\gamma xy/R^5.$$

p, q and r are the angular velocities about the triad formed by the axes of inertia of the Moon. α, β, γ are defined as usual by

$$\alpha = (C-B)/A, \quad \beta = (C-A)/B, \quad \gamma = (B-A)/C,$$

A, B and C are the moments of inertia of the Moon, A the least and C the greatest. The A axis points approximately to the Earth, the C axis is approximately normal to the orbit of the Moon.

The right sides are the components of the couple

$$M\boldsymbol{r} \wedge \operatorname{grad} V,$$

where $\boldsymbol{r}$ is the position vector of the Earth referred to the triad of the principal axes of inertia of the Moon and V is the potential energy of the Earth in the gravitational field of the Moon, namely

$$-\tfrac{3}{2}(M/R^5)\,(Ax^2+By^2+Cz^2)$$

to second order; M is the mass of the Earth and R the distance of the Earth from the Moon.

Hayn expresses x, y, z in terms of L, the true longitude of the Moon, B the latitude of the Moon and R, and obtains L, sin B and R from Hansen's tables of the Moon, which however give the coefficients of the various periodic terms as numbers. Thus Hayn's theory is semi-numerical. His procedure works well for an elementary theory when only a few terms are included in the torques and retained in the solutions, but becomes cumbersome when the calculations are to be continued to include more terms in the torques and higher orders of terms in the solution and when nonlinear interactions are to be included. One of the sources of difficulty is that the torques are the components of the vector product of the gradient of the lunar optical potential by the radius vector of the disturbing body; the manipulation would be simplified if vector multiplication could be avoided. Nonetheless Eckhardt (1970, 1973) has based his semi-literal theory upon Euler's equations as have Kaula & Baxa (1973).

The complications in the theory of libration arise in three ways, of which the most fundamental is that the equations of motion are nonlinear. Consider a system of axes centred in the Moon and rotating at the mean angular velocity of the Moon. The physical librations may then be described by the angular displacements of the principal axes of inertia of the Moon from that frame in uniform rotations. The torques exerted on the Moon by the Earth are functions of the direction cosines of the Earth relative to the principal axes of inertia of the Moon and the direction of the Earth relative to those axes is the resultant of the direction of the Earth relative to the axes in uniform rotation, which is known from the theory of the orbital motion of the Moon, compounded with the rotation of the Moon relative to the same axes. Thus the torques are functions not just of the direction cosines of the Earth but also of the variables which describe the rotation of the Moon and which are the unknowns to be calculated. In consequence the equations of motion are non-linear.

The second problem concerns the way in which the angular position of the Moon and the direction of the Earth are specified. The general practice is to use the angles, θ, ϕ, ψ, where θ is the inclination of the Moon's equator to the ecliptic, ϕ is the angle between the axis of least moment of inertia and the ascending node of the equator of the Moon on the ecliptic, and ψ is the longitude of the lunar node, but ϕ is not a small angle although the corresponding velocity and acceleration are small. The direction of the Moon is given in the usual form by the longitude, L, and latitude B and the radius vector, R, each of which is expressed as a Fourier series. It turns out that calculations in terms of these variables is rather cumbersome, and the lunar longitude, latitude and radius vector are not the quantities directly calculated in Brown's lunar theory. Brown's primary expressions are for the Cartesian coordinates of the Moon in a frame rotating about the Earth at the speed $(n-n_0)$ where n is the mean motion of the Moon and n_0 that of the Sun. There is a considerable gain in clarity if the direction cosines of the Earth from the Moon are derived directly from Brown's primary Cartesian coordinates and if at the same time it is arranged that the variables that describe the rotation of the Moon are all small quantities.

The third difficulty with Euler's equation lies in devising a simple systematic scheme for the calculation of the torques.

Euler's equations have formed the basis of three theories, the semi-algebraic one of Hayn (1902, 1923) which is not carried to a sufficiently high order for application to lunar ranging, that of Eckhardt (1970, 1973), already mentioned, in which again the coefficients of the periodic terms are calculated numerically from numerical values for the dynamical parameters of the Moon and the ephemeris of her orbit about the Earth, and that of Kaula & Baxa (1973), who developed a literal theory to include the harmonic terms of third order in the potential of the Moon. Hayn's & Eckhardt's theories both suffer from the disadvantage that they do not display explicitly how the important parameters α, β, γ enter the coefficients of the various periodic terms in the librations.

In view of the great convenience of the Hamiltonian form of the equations of motion in celestial mechanics, it is natural to ask how useful it would be for lunar librations. Jönsson (1917) has developed a theory using the Hamiltonian equations but it does not seem that any substantial advantage occurs corresponding to that that comes from the use of the von Zeipel or Lie transform reductions in orbital theory – the reason is no doubt that the arguments of periodic terms do not form well defined groups of distinct orders of magnitude. Jeffreys (1955, 1961) on the other hand, showed that notable simplifications were effected in two ways – the use of the equations of motion in Lagrangian form and the use of Cartesian coordinates to express the position of the Earth (or other disturbing body) relative to the Moon together with a system of angular coordinates for the rotation of the Moon which are always small. Further study has confirmed these advantages and has shown that Jeffreys's procedure, with small modifications, lends itself well to algebraic manipulation by computer; it also enables Brown's (1899) expressions for the Cartesian coordinates of the Moon to be used directly to calculate those of the Earth relative to the Moon.

3. Outline of the theory of libration using Lagrange's equations of motion

In preparation for a discussion of the use of a program for algebraic manipulation by computer, a brief outline of the Lagrangian theory is given in this section; for simplicity only the principal term in the libration is calculated.

The first step is to define the geometrical framework.

To a first approximation the Earth moves in a circular orbit about the Moon and the Moon rotates at the same speed about its axis. Thus the Earth maintains a constant direction relative to axes fixed in the Moon. Further, according to Cassini's rule, the ecliptic, the plane of the Moon's orbit and the equator of the Moon all intersect in the same line; in practice however the Earth's orbit is elliptical and Cassini's rules are not followed strictly.

In figure 1, XY is the plane of the ecliptic and Z is the pole of the ecliptic. The axes $OXYZ$ rotate about OZ at the uniform rate n which is the average value of the mean motion of the Earth about the Moon. OX is in the approximate direction of the Earth; OY is approximately tangential to the Moon's orbit. 0123 coincide with the Moon's principal moment of inertia, as follows:

$$O3\colon C,\quad O2\colon B,\quad O1\colon A.$$

(For stability, $C > B > A$.)

Let y_1, y_2 and y_3 be rotations which relate the two systems of axes:

$$\begin{aligned}
&y_1 \text{ about } OZ\colon && OX \rightarrow OX', \\
& && OY \rightarrow OY'; \\
&y_2 \text{ about } OY'\colon && OX' \rightarrow O1, \\
& && OZ \rightarrow OZ'; \\
&y_3 \text{ about } O1\colon && OY' \rightarrow O2, \\
& && OZ' \rightarrow O3.
\end{aligned}$$

The angular velocities are:

	$n+\dot{y}_1$	about OZ
giving	$(n+\dot{y}_1)\cos y_2$	about OZ',
	$-(n+\dot{y}_1)\sin y_2$	about $O1$,
	$(n+\dot{y}_1)\cos y_2 \sin y_3$	about $O2$,
and	$(n+\dot{y}_1)\cos y_2 \cos y_3$	about $O3$;
	$\dot{y}_2$	about OY
giving	$\dot{y}_2 \cos y_3$	about $O2$,
and	$\dot{y}_2 \sin y_3$	about $O3$;
and finally	$\dot{y}_3$	about $O1$.

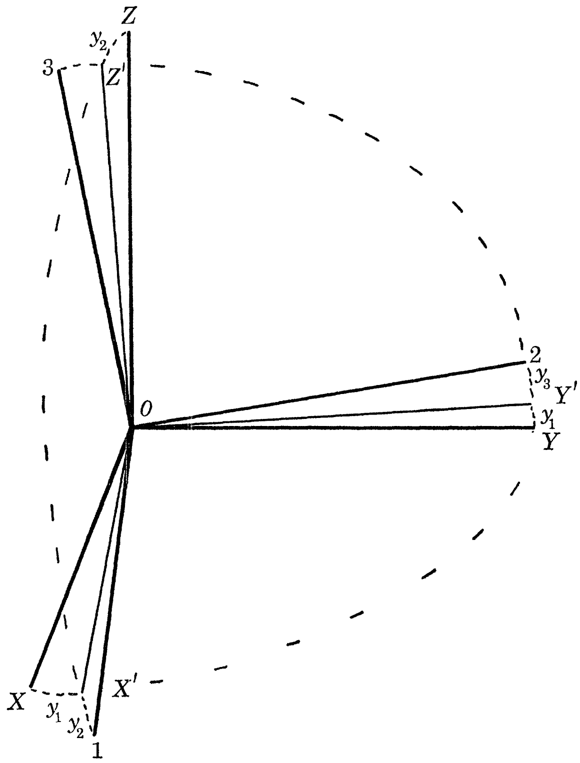

FIGURE 1. Geometry of the principal axes of the Moon and their rotations relative to axes in uniform rotation.

The overall velocities are thus

$$\begin{aligned} \text{about } O1\text{:} \quad & -(n+\dot{y}_1)\sin y_2+\dot{y}_3 = \Omega_1, \\ O2\text{:} \quad & (n+\dot{y}_1)\cos y_2 \sin y_3+\dot{y}_2\cos y_3 = \Omega_2, \\ O3\text{:} \quad & (n+\dot{y}_1)\cos y_2 \cos y_3-\dot{y}_2\sin y_3 = \Omega_3. \end{aligned}$$

The kinetic energy, T is

$$\tfrac{1}{2}(A\Omega_1^2+B\Omega_2^2+C\Omega_3^2).$$

If l_1, l_2, l_3 are the direction cosines of the Earth relative to the principal axes of the Moon, the potential energy V is given to second order by

$$-\frac{GM}{2r^3}[A+B+C-3(Al_1^2+Bl_2^2+Cl_3^2)],$$

where M is the mass of the Earth and its distance from the Moon.

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If l_1', l_2', l_3' are the direction cosines of the Earth from the Moon in the ecliptic system, $OXYZ$ (as obtained from Brown's theory of the Moon's orbit) then

$$\boldsymbol{l} = \boldsymbol{R}\cdot\boldsymbol{l}',$$

where the rotation matrix, $\boldsymbol{R}$, is the product of the three matrices.

$$\boldsymbol{R}_1 = \begin{pmatrix} \cos y_1 & \sin y_1 & \cdot \\ -\sin y_1 & \cos y_1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix};$$

$$\boldsymbol{R}_2 = \begin{pmatrix} \cos y_2 & \cdot & -\sin y_2 \\ \cdot & 1 & \cdot \\ \sin y_2 & \cdot & \cos y_2 \end{pmatrix};$$

$$\boldsymbol{R}_3 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cos y_3 & \sin y_3 \\ \cdot & -\sin y_2 & \cos y_3 \end{pmatrix}.$$

$\boldsymbol{l}'$ is the unit vector in the direction of the Earth and may be found most directly from E. W. Brown's lunar theory if the OXY plane is taken to be the XY plane of Brown's theory.

Brown takes axes rotating at n_0 instead of n, where n_0 is the mean motion of the Sun about the Earth. The rectangular coordinates of the Moon are obtained in axes with the Earth at the origin and the components of the vector $\boldsymbol{l}'$ are obtained by changing signs.

The leading terms are:

$$l_1' = 1$$
$$l_2' = 0$$
$$l_3' = -2k\sin\nu t,$$

where $k = \sin\frac{1}{2}i$, i is the inclination of the Moon's orbit to the ecliptic and $\nu = (n-n_0)\,g$.

g is slightly different from 1 and is determined by the motion of the node of the Moon's orbit upon the ecliptic. Then

$$\boldsymbol{l} = \begin{pmatrix} \cos y_1 \cos y_2 - 2k\sin y_2 \sin\nu t \\ -\sin y_1 \cos y_3 + \cos y_1 \sin y_2 \sin y_3 + 2k\cos y_2 \sin y_3 \sin\nu t \\ \sin y_1 \sin y_3 + \cos y_1 \cos y_2 \cos y_3 + 2k\cos y_2 \cos y_3 \sin\nu t \end{pmatrix}.$$

Because T depends on y_i and $\dot{y}_i$ but V on y_i only, the equations of motion in Lagrangian form read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial y_i} - \frac{\partial T}{\partial y_i} = \frac{\partial V}{\partial y_i}.$$

In forming the differentials, it may be supposed that y_i is proportional to $\mathrm{e}^{i\nu t}$ because the force function is proportional to $\sin\nu t$.

Thus

$$\dot{y}_i = i\nu y_i; \quad \ddot{y}_i = -\nu^2 y_i.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}_i} = \begin{pmatrix} -C\nu^2 y_1 \\ -B\nu^2 y_2 + (B-C)\,in\nu y_3 \\ -A\nu^3 y_3 - Ain\nu \dot{y}_2 \end{pmatrix}$$

and

$$\frac{\partial T}{\partial y_i} = \begin{pmatrix} 0 \\ (A-C)\,n^2 y_2 - in\nu A y_3 \\ in\nu(B-C)\,y_2 + (B-C)\,n^2 y^3 \end{pmatrix}.$$

Thus
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{y}_1}-\frac{\partial T}{\partial y_1}=\begin{pmatrix}-C\nu^2 & \cdot & \cdot \\ \cdot & -B\nu^2+(C-A)\,n^2 & (B-C+A)\,\mathrm{i}n\nu \\ \cdot & (-A-B+C)\,\mathrm{i}n\nu & -A\nu^2+(C-B)\,n^2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}.$$

At this stage it is convenient to simplify the equations by dividing the first line by C, the second by B and the third by A, and to normalize the speeds by setting $\nu = n\nu$ and dividing throughout by n^2. The vector then reads

$$-\begin{pmatrix}\nu^2 & \cdot & \cdot \\ \cdot & \nu^2-\beta & -\mathrm{i}\nu(1-\beta) \\ \cdot & \mathrm{i}\nu(1-\alpha) & \nu^2-\alpha\end{pmatrix}\begin{pmatrix}Cy_1\\ By_2\\ Ay_3\end{pmatrix},$$

where $\alpha = (C-B)/A$ and $\beta = (C-A)/B$.

In the potential, $GM/r^3 = n^2$ to first order (in the second order, GM/r^3 differs slightly from n because of the attraction of the Sun on the Moon). Thus

$$\frac{\partial V}{\partial y_1} = -3n^2\begin{pmatrix}(B-A)\,y_1\\ (C-A)\,y_2+2(C-A)\,k\sin\nu t\\ 0\end{pmatrix}.$$

Again divide by C, B, A respectively and put $n = 1$ and $(B-A)/C = \gamma$:

$$\frac{\partial V}{\partial y_1} = -3\begin{pmatrix}\gamma\\ \beta\\ 0\end{pmatrix}\begin{pmatrix}Cy_1\\ By_2\\ Ay_3\end{pmatrix}-\begin{pmatrix}0\\ 6B\beta k\sin\nu t\\ 0\end{pmatrix}.$$

Now since $\sin\nu t = 1/2\mathrm{i}(\mathrm{e}^{\mathrm{i}\nu t}-\mathrm{e}^{-\mathrm{i}\nu t})$ it will be seen that y_i are sums of terms proportional to $\mathrm{e}^{\mathrm{i}\nu t}$ and those proportional to $\mathrm{e}^{-\mathrm{i}\nu t}$. Thus the Lagrangian equations may be written

$$\boldsymbol{M}\cdot\boldsymbol{y} = -\frac{3\beta k}{\mathrm{i}}\begin{pmatrix}0\\ \mathrm{e}^{\pm\mathrm{i}\nu t}\\ 0\end{pmatrix},$$

where
$$\boldsymbol{M} = -\begin{pmatrix}\nu^2-3\gamma & \cdot & \cdot \\ \cdot & \nu^2-4\beta & -\mathrm{i}\nu(1-\beta) \\ \cdot & \mathrm{i}\nu(1-\alpha) & \nu^2-\alpha\end{pmatrix}$$

and the rows have been divided by C, B, A respectively, and the terms proportional to y_1 and y_2 in $-\partial V/\partial y_1$ and $-\partial V/\partial y_2$ have been taken into $\boldsymbol{M}\cdot\boldsymbol{y}$.

In an elementary solution, α, β and γ, which are of order 10^{-3}, may be ignored on the left side in comparison with ν but not in comparison with ν^2-1.

To that approximation

$$\boldsymbol{M}^{-1} = -\begin{pmatrix}1/\nu^2 & \cdot & \cdot \\ \cdot & (\nu^2-1-3\beta)^{-1} & \mathrm{i}\nu^{-1}(\nu^2-1-3\beta)^{-1} \\ \cdot & -\mathrm{i}\nu^{-1}(\nu^2-1-3\beta)^{-1} & (\nu^2-1-3\beta)^{-1}\end{pmatrix}$$

it being understood that ν stands for $+\nu$ when the forcing function is proportional to $\mathrm{e}^{\mathrm{i}\nu t}$, and that it is to be replaced by $-\nu$ when the forcing function is proportional to $\mathrm{e}^{-\mathrm{i}\nu t}$.

Thus
$$\boldsymbol{y} = -\frac{3k\beta}{\mathrm{i}(\nu^2-1-3\beta)}\begin{pmatrix}0\\ \mathrm{e}^{\mathrm{i}\nu t}-\mathrm{e}^{-\mathrm{i}\nu t}\\ -\mathrm{i}(\mathrm{e}^{\mathrm{i}\nu t}+\mathrm{e}^{-\mathrm{i}\nu t})\,\nu^{-1}\end{pmatrix}.$$

Now the normalized ν is $g(n-n_0)/n$ that is $1+g'-n_0/n$, where $g' = 0.0852$ and $n_0/n = 0.0808$. Accordingly there is no forced oscillation of y_1 but

$$y_2 = \frac{6k\beta \sin \nu t}{2(g'-n_0/n)-3\beta}$$

and

$$y_3 = \frac{6k\beta \cos \nu t}{\nu[2(g'-n_0/n)-3\beta]}.$$

The solutions represent a rotation of the C axis about the pole of the ecliptic at the speed of $(n-n_0)(1+g)$ and with the amplitude $6k\beta[2(g'-n_0/n)-3\beta]$, in other words the C axis is inclined to the ecliptic at the angle

$$\frac{6k\beta}{2(g'-n_0/n)-3\beta}$$

(the motion is not strictly in a circular cone because y_2 and y_3 differ by the factor $\nu-1$, which is however very close to unity). With $k = 0.045$, $\beta = 6\times 10^{-4}$ and $g'-n_0/n = 0.005$, the angle is about $1^\circ\,30'$.

It will be seen that the factor $1/(\nu^2-1-3\beta)$ results in a large amplification when ν is close to 1 but for those speeds (the majority) for which ν is not close to 0 or 1, there is no comparable amplification.

Another simple term arises from the eccentricity of the Moon's orbit. The direction cosines of the Earth are, to first order in the eccentricity, e, of the orbit

$$l_1' = 1,$$
$$l_2' = 2e \sin \nu' t,$$
$$l_3' = 0,$$

where $\nu' = n-n_0-c$, and c is the motion of perigee of the Moon's orbit, and the solutions are approximately

$$y_1 = 3\gamma e \sin \nu' t,$$
$$y_2 = 0,$$
$$y_3 = 0;$$

the motion is known as the *libration in longitude* since it is a rotation about the C-axis of the Moon. A larger term in the libration in longitude has the speed of the Sun's motion around the Earth and the Moon.

For the interpretation of lunar laser ranging, the theory must include terms of higher order than α^2, β^2. This means

(1) Terms of that order must be kept in the coefficients of y_1 and terms of order α^3, β^3 in the right hand forcing functions.

(2) Correspondingly many more terms of Brown's theory must be retained in the direction cosines of the Earth.

The main problem is the nonlinearity of the equation of motion on the left sides, terms like $y_1 y_k$ and $y_1 \ddot{y}_k$ occur, while in the forcing functions, there are terms like $k y_1 \sin \nu t$.

Finally, additional terms must be included in the potential, namely the attraction of the Sun and the third and fourth harmonics in the lunar potential, which are quite significant.

Some of these problems have already been dealt with (Jeffreys 1961; Eckhardt 1973). The manipulation is sufficiently arduous for a linear theory (see Kaula & Baxa 1973; Jönsson 1917) and for adequate precision it is almost essential to use machine symbol manipulation for the algebraic development.

4. Algebraic manipulation by computer

I carried out some preliminary work on machine manipulation for a literal theory of librations with the Formac system maintained on the Edinburgh Regional Computer but I have done the major part with the Camal system developed and maintained at the Cambridge Computer Laboratory. Camal offers a number of important advantages over Formac in handling the problems arising in the theory of librations. Briefly, Camal in common with other systems for

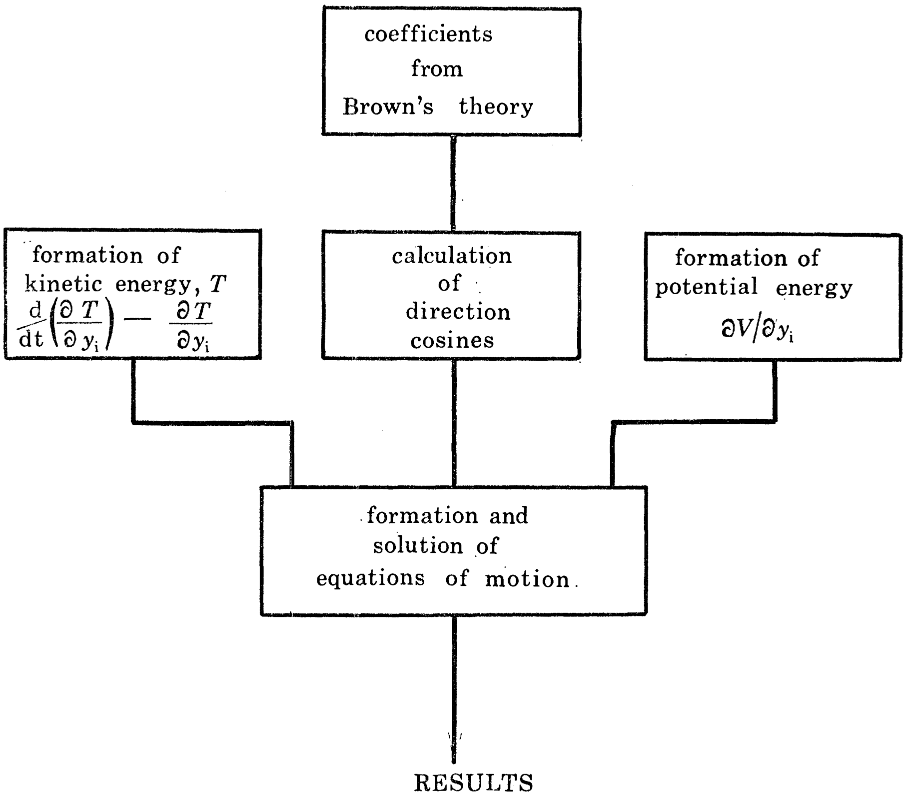

Figure 2. Flow diagram of algebraic manipulations by computer.

algebraic manipulations provides the means of deriving algebraic entities from other algebraic entities by the operation of algebra, by differentation and, to a certain extent, integration, by substitution and by numerical evaluation. The algebraic entities may be polynomials, Fourier series or series of complex exponentials. Facilities are available for controlling the order of terms retained in operations, for editing entities and for extracting sub-entities with particular properties. In conjunction with the filing system operated at Cambridge, entities may be filed for use in subsequent calculations and may be extracted from the files in those calculations. The latter facility enables computations to be split into sections which both enable sections to be developed and run independently and also cuts down the requirements for store during the running of a section.

If attention is confined initially to the linearized equations of motion from which all terms of second and higher order in the y_i are omitted, the formation and solution of the equations has been divided into four stages as follows (figure 2):

(1) Calculation of the direction cosines and reciprocal distance of the Earth from the Moon from Brown's lunar theory.

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(2) Formation and differentation of the kinetic energy, up to terms of first order in y_i.

(3) Formation and differentation of the potential energy, up to terms of first order in y_i.

(4) Formation of the Lagrangian equations of motion, reduction of them to algebraic equations by assuming solutions proportional to exp $(i\nu t)$ and solution matrix inversion for each separate term in the forcing function.

The results of the first three stages are written into the files; in the final stage those files are read and the results are written into another file.

The solutions of the linearized equations of motion are not adequate, as may be seen from the fact that the largest terms of the solutions corresponding to the inclinations of the Moon's C-axis to the ecliptic, are of order 10^{-2}; it is thus evident that terms of the order of the square and cube of the linearized solutions must be kept in the final solution, and so corresponding nonlinear terms must be retained in the equations of motion. The differentation of both the kinetic and potential energy give rise to terms with the squares and higher powers of the unknown y_i; the procedure is to substitute the linearized solutions for the y_i; calculate the nonlinear terms and treat the result as the forcing function for the equations of motion, now to be solved for corrections to the linearized solutions. In practice it is convenient to determine the corrections in a number of stages.

Two problems present some difficulty in the manipulation, the treatment of the small divisions and the treatment of constant terms. In each case the reason for the difficulty is that the only form of division available in CAMAL is by the binomial theorem,

$$(1+x)^{-1} = 1-x+x^2....$$

Now the inverse matrix of the equations of motion is of the form

$$A/D,$$

where A is a matrix of terms that are polynomials in ν, α, β, e, k, and so on while D, the determinant is a similar polynomial.

When $\nu-1$ is not small, the form of D and of the elements of A is such that they can all be reduced in a straightforward manner to the form

$$N(1+x),$$

where N is a numerical factor and x, a polynomial in $\alpha, e, \ldots$, is small compared with 1.

That cannot be done with small divisors such as

$$\{2(g'-n_0/n)-3\beta\};$$

the procedure I have adopted is to write g', n_0/n and β in the form $\eta_0+\eta'$, where η_0 is an approximate *numerical* value and η' is a literal residual.

I use the same procedure for constant terms which for example arise, as shown by Eckhardt (1973) and Kaula & Baxa (1973), from harmonics of third order in the lunar potential. ν is zero for constant terms and the inverse matrix is

$$\begin{pmatrix} 1/3\gamma & \cdot & \cdot \\ \cdot & 1/4\beta & \cdot \\ \cdot & \cdot & 1/\alpha \end{pmatrix}.$$

Again, it is necessary to substitute for the literal quantity α, β, or γ, a numerical approximation plus a literal residual.

5. Nonlinear effects

Nonlinear interactions primarily generate terms of order α^2, $\alpha\beta$ and so on, multiplied by e, k, etc., but in addition they lead to two particular types of behaviour. It can readily be seen that the equation of motion for the libration in longitude is nonlinear, involving a term in y_1^3, and there has been some discussion of the behaviour. However, it is unrealistic to write down a separate equation for y_1 independent of y_2 and y_3, for although to first order the equation for y_1 is independent of the other two, as may be seen from the form of the elementary equations of motion, so soon as nonlinear terms are included, the independence breaks down.

The second question is that of the so-called free-librations, the periods of which are obtained by solving the determinantal equation

$$|\boldsymbol{M}| = 0$$

or

$$(\nu^2-3\gamma)\{\nu^4-\nu^2(1+3\beta+\alpha\beta)+4\alpha\beta\},$$

the solutions of which are $\pm(3\gamma)^{\frac{1}{2}}$, $\pm(1+3\beta)^{\frac{1}{2}}$ and $\pm(2\alpha\beta)^{\frac{1}{2}}$.

The motion with speed $n(3\gamma)^{\frac{1}{2}}$ has been much discussed and long sought (see Jeffreys 1961) but because the equations of motion are non-linear any 'free' motion will be coupled to, and excited by, forced motions, so that it cannot be said to be free in the sense that complimentary functions of a set of linear differential equations represent motions independent of the forced motions.

6. Conclusions

From the large number of terms of order (e, k) (α, β, γ) that make up the forced solutions of the equations of the librations, four terms or groups of terms are worth particular notice. In the first place, there are those which are an order of magnitude larger than most because of a small division in the solutions, in particular, the terms which depend on the inclination of the orbit to the ecliptic and correspond to the inclination of the C-axis to the ecliptic. There are also such terms, though smaller, in the libration in longitude. Secondly, there are constant terms, which arise for example when the potential contains third order harmonics. Because the coefficients of those harmonics in the Moon's potential are relatively large, the constant terms are also significant, at least to the accuracy with which lunar laser ranging is conducted.

Thirdly, there are nonlinear terms shown in particular by the libration in longitude, although significant in all variables.

Lastly, there are the free librations, the independent existence of which requires further theoretical discussion and which are still to be found by observation.

All the foregoing remarks cover the librations forced by the motion of the Earth in the gravitational field of the Moon, but the effect of the Sun rotating about the Moon is not negligible at the accuracy of lunar laser ranging; its inclination is however quite straightforward, simply requiring the addition of extra terms in the potential.

Suppose a lunar retroreflector is at a site the direction cosines of which relative to the Moon's axes of inertia are of the order of $\frac{1}{2}$. The effect of a term of speed ω and amplitude δ in any of the librations will be a variation in the distance of the reflector from a terrestrial observatory of order

$$\tfrac{1}{2}\delta r_M \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} \omega t,$$

where r_M is the radius of the Moon; ω is in general of the form

$$(\mu_1 n + \mu_2 n_0 + \mu_3 g + \mu_4 c),$$

where μ_1, μ_2, μ_3 and μ_4 are integers and n, n_0, g and c have already been defined. The slowest speeds will have μ_1 equal to zero, but most terms have speeds which are multiples of n, the motion of the Moon about the Earth.

δ is of order, as has been said, (e, k) (α, β) or less, with the exception of terms with small divisors. e and k are of order 10^{-1}, α, β of order 5×10^{-4}, r_M is 1700 km, thus the typical term in the physical libration leads to a change of 40 m or less in the distance to a retroreflector.

The principal inclination terms, however, correspond to a variation of some 25 km (in mid-latitudes). If it be supposed that measurements will be made with precisions of 5 cm or better, then terms of order (e^2, k^2) (α, β) and (e, k) (α^2, β^2) should be retained, but not (e^2, k^2) (α^2, β^2) unless there are small divisors.

These considerations raise the question of whether neglected terms may accumulate to a significant degree, bearing in mind the rapid increase in the numbers of terms with the increasing order of the polynomial coefficients. CAMAL provides good facilities for selecting terms of specified order and sub-routines have been written to take into account the magnitudes of numerical coefficients, but as often with empirical developments in series, there is no assured way of setting bounds to neglected terms.

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Discussion

J. KOVALEVSKY (*C.E.R.G.A.* 8 *Boulevard Emile Zola* – 06130 – *Grasse, France*). A purely literal theory of the libration of the Moon is being derived by A. Migus, in Paris, using the Poisson series precessor (Deprit–Henrard). In this work, he uses the periodic parts of the departures from the three resonant conditions as canonical variables of the problem. The equations are solved by a method of successive approximations.

Brown's lunar theory is used and all terms of the main problem, larger than 1″ in longitude and latitude and 0.015″ in parallax are kept. Third harmonics as well as relative corrections to the second harmonics of the lunar gravitational field are included in a literal form. In a second stage, now in progress, direct solar effects, planetary terms, fourth harmonics and third order terms of the Hamiltonian are being included.

Comparing the series already obtained with Eckhardt's results, a very good agreement for most of the terms has been found. The larger differences refer to the constant terms in τ (4.1″) and p_1 (0.5″). Four other terms differ by more than 0.1″ and only five in each series have differences larger than 0.01″.

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